

Purity for overconvergence

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Abstract

Let $X \hookrightarrow \overline{X}$ be an open immersion of smooth varieties over a field of characteristic $p > 0$ such that the complement is a simple normal crossing divisor and let $\overline{Z} \subseteq Z \subseteq \overline{X}$ be closed subschemes of codimension at least 2. In this paper, we prove that the canonical restriction functor between the category of overconvergent F -isocrystals $F\text{-Isoc}^\dagger(X, \overline{X}) \longrightarrow F\text{-Isoc}^\dagger(X \setminus Z, \overline{X} \setminus \overline{Z})$ is an equivalence of categories. We also prove an application to the category of p -adic representations of the fundamental group of X , which is a higher-dimensional version of a result of Tsuzuki.

Contents

1	Preliminaries	3
2	Generic convergence and overconvergence	5
3	Purity for overconvergence	9
4	An application	18

Introduction

Let X be a regular scheme and let $Z \subseteq X$ be a closed subscheme of codimension at least 2. Then, by the famous Zariski-Nagata purity, any locally constant constructible sheaf on $(X \setminus Z)_{\text{et}}$ extends uniquely to a locally constant constructible sheaf on X_{et} . As a p -adic analogue of this fact, Kedlaya proved in [11, 5.3.3] the following result on the purity for overconvergent isocrystals: Let k be a field of characteristic $p > 0$, let $X \hookrightarrow \overline{X}$ be an open immersion of k -varieties with X smooth and let $Z \subseteq X$ be a closed subscheme of X of codimension at least 2 such that

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$X \setminus Z$ is dense in \overline{X} . Then the restriction functor of the categories of overconvergent isocrystals

$$\mathrm{Isoc}^\dagger(X, \overline{X}) \longrightarrow \mathrm{Isoc}^\dagger(X \setminus Z, \overline{X})$$

is an equivalence of categories.

In this paper, we prove a slight generalization of the result of Kedlaya in the case where \overline{X} is smooth, $\overline{X} \setminus X$ is a simple normal crossing divisor and the overconvergent isocrystals are endowed with Frobenius structure. The precise statement is as follows: Let $X \hookrightarrow \overline{X}$ be an open immersion of smooth varieties over a field of characteristic $p > 0$ such that the complement is a simple normal crossing divisor and let $\overline{Z} \subseteq Z \subseteq \overline{X}$ be closed subschemes of codimension at least 2. Then the canonical restriction functor between the category of overconvergent F -isocrystals

$$F\text{-Isoc}^\dagger(X, \overline{X}) \longrightarrow F\text{-Isoc}^\dagger(X \setminus Z, \overline{X} \setminus \overline{Z})$$

is an equivalence of categories. In the case $\overline{Z} = \emptyset$, this is reduced to the above-mentioned result of Kedlaya. A new point in our result is that we have the purity also on the second factor of the pair (X, \overline{X}) (the overconvergent locus). In this sense, we can say our result as ‘the purity for overconvergence’.

As an application, we consider the following. Assume that k is perfect and that X is connected. Then Crew proved the equivalence of categories

$$G : \mathrm{Rep}_{K^\sigma}(\pi(X)) \xrightarrow{\cong} F\text{-Isoc}(X)^\circ$$

between the category $\mathrm{Rep}_{K^\sigma}(\pi(X))$ of finite-dimensional continuous representations of the fundamental group $\pi_1(X)$ of X over K^σ (where K is a complete discrete valuation field with residue field k endowed with a lifting σ of Frobenius and K^σ denotes the fixed field of σ) and the category $F\text{-Isoc}(X)^\circ$ of unit-root convergent F -isocrystals on X over K . When X is a curve, Tsuzuki proved in [18] that the subcategory $F\text{-Isoc}^\dagger(X, \overline{X})^\circ$ of $F\text{-Isoc}(X)^\circ$ consisting of unit-root overconvergent F -isocrystals on (X, \overline{X}) over K corresponds (via G) to the subcategory of $\mathrm{Rep}_{K^\sigma}(\pi(X))$ consisting of ‘the representations with finite local monodromy’. A generalization of this result to higher-dimensional case is proved by Kedlaya in [13, 2.3.7, 2.3.9] based on a result of Tsuzuki in [19]. One drawback in Kedlaya’s result is that the number of valuations which we should look at is infinite. In this paper, we give an alternative definition of ‘the representations with finite local monodromy’ which looks at only finitely many discrete valuations of $k(X)$ and prove that the category of such representations is still equivalent (via G) to $F\text{-Isoc}(X, \overline{X})^\circ$, by using the purity for overconvergence mentioned in the previous paragraph.

The content of each section is as follows: In the first section, we give several notations, conventions and terminologies which we often use in this paper. In the second section, we prove a theorem on the relation of generic convergence and overconvergence of p -adic differential equations in certain case, which we need to prove the purity for overconvergence. In the third section, we give a proof of the purity for overconvergence. In the fourth section, we explain the above-mentioned application to p -adic representations.

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1 Preliminaries

In this section, we give several notations, conventions and terminologies which we often use in this paper.

Throughout this paper, K is a complete discrete valuation field of mixed characteristic $(0, p)$ with ring of integers O_K and residue field k . Let q be a fixed power of p and assume that k contains \mathbb{F}_q . Moreover, let $\sigma : O_K \rightarrow O_K$ be an endomorphism of O_K which lifts the q -th power map on k . We denote the endomorphism on K induced by σ by the same symbol. Let $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$ be a fixed valuation of K and let Γ^* be $\sqrt{|K^\times|} \cup \{0\}$.

All the schemes appearing in this paper are assumed to be separated of finite type over k and all the p -adic formal schemes appearing in this paper are assumed to be separated, topologically of finite type over $\mathrm{Spf} O_K$. For a p -adic formal scheme \mathcal{X} , we call the scheme $\mathcal{X} \otimes_{O_K} k$ the special fiber of \mathcal{X} . For a smooth scheme X over k , a lift of X is a closed immersion $X \hookrightarrow \mathcal{X}$ into a smooth p -adic formal scheme \mathcal{X} such that X is naturally isomorphic to the special fiber of \mathcal{X} . (Note that, when X is affine and smooth over k , a lift of X always exists.) For a p -adic formal scheme \mathcal{X} , we denote the associated rigid space over K by \mathcal{X}_K . Then we have the specialization map $\mathrm{sp} : \mathcal{X}_K \rightarrow \mathcal{X}$.

A pair (X, \overline{X}) is a pair of schemes X, \overline{X} (separated of finite type over k) endowed with an open immersion $X \hookrightarrow \overline{X}$ over k . A smooth pair is a pair (X, \overline{X}) such that X, \overline{X} are smooth and that $\overline{X} \setminus X$ (endowed with the reduced closed subscheme structure) is a simple normal crossing divisor in \overline{X} . A formal pair $(\mathcal{X}, \overline{\mathcal{X}})$ is a pair of p -adic formal schemes $\mathcal{X}, \overline{\mathcal{X}}$ (separated, topologically of finite type over $\mathrm{Spf} O_K$) endowed with an open immersion $\mathcal{X} \hookrightarrow \overline{\mathcal{X}}$ over $\mathrm{Spf} O_K$. A formal smooth pair is a pair $(\mathcal{X}, \overline{\mathcal{X}})$ such that $\mathcal{X}, \overline{\mathcal{X}}$ are smooth and that $\overline{\mathcal{X}} \setminus \mathcal{X}$ (with some closed formal subscheme structure) is a relative simple normal crossing divisor in $\overline{\mathcal{X}}$. A morphism of (formal) pairs $f : (X, \overline{X}) \rightarrow (Y, \overline{Y})$ is defined as the morphism $f : \overline{X} \rightarrow \overline{Y}$ satisfying $f(X) \subseteq Y$. It is called strict if $f^{-1}(Y) = X$ and it is called finite etale or a closed immersion if so is $f : \overline{X} \rightarrow \overline{Y}$.

For a formal pair $(\mathcal{X}, \overline{\mathcal{X}})$, we call the pair $(\mathcal{X} \otimes_{O_K} k, \overline{\mathcal{X}} \otimes_{O_K} k)$ the special fiber of $(\mathcal{X}, \overline{\mathcal{X}})$. For a smooth pair (X, \overline{X}) , a lift of (X, \overline{X}) is a strict closed immersion $(X, \overline{X}) \hookrightarrow (\mathcal{X}, \overline{\mathcal{X}})$ into a formal smooth pair such that (X, \overline{X}) is naturally isomorphic to the special fiber of $(\mathcal{X}, \overline{\mathcal{X}})$.

Let $(\mathcal{X}, \overline{\mathcal{X}})$ be a formal smooth pair. Then a strict neighborhood of \mathcal{X}_K in $\overline{\mathcal{X}}_K$

is an admissible open set V of $\overline{\mathcal{X}}_K$ such that $\{V, \text{sp}^{-1}(\overline{\mathcal{X}} \setminus \mathcal{X})\}$ forms an admissible covering of $\overline{\mathcal{X}}_K$. For strict neighborhoods V, W of \mathcal{X}_K in $\overline{\mathcal{X}}_K$ with $W \subseteq V$, we denote the canonical open immersion $W \hookrightarrow V$ by α_{WV} and using this, we define the sheaf of overconvergent sections $j^\dagger \mathcal{O}_{\overline{\mathcal{X}}_K}$ by $j^\dagger \mathcal{O}_{\overline{\mathcal{X}}_K} := \varinjlim_V \alpha_{V\overline{\mathcal{X}}_K,*} \mathcal{O}_V$. For a strict neighborhood V of \mathcal{X}_K in $\overline{\mathcal{X}}_K$ and an \mathcal{O}_V -module E , we put $j_V^\dagger E := \varinjlim_{W \subseteq V} \alpha_{W\overline{\mathcal{X}}_K,*} \alpha_{WV}^{-1} E$. Let $\text{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K)$ be the category of pairs $(V, (E, \nabla))$ consisting of a strict neighborhood V of \mathcal{X}_K in $\overline{\mathcal{X}}_K$ and a ∇ -module (=locally free module of finite rank endowed with an integrable connection) (E, ∇) on V over K , whose set of morphisms is defined by $\text{Hom}((V, (E, \nabla)), (V', (E', \nabla'))) := \varinjlim_{V''} \text{Hom}((E, \nabla)|_{V''}, (E', \nabla')|_{V''})$, where V'' runs through strict neighborhoods of \mathcal{X}_K in $\overline{\mathcal{X}}_K$ contained in $V \cap V'$. We call an object in $\text{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K)$ a ∇ -module on a strict neighborhood of \mathcal{X}_K in $\overline{\mathcal{X}}_K$ by abuse of terminology, and we will often denote it simply by (E, ∇) in the following. On the other hand, let $\text{MIC}(j^\dagger \mathcal{O}_{\overline{\mathcal{X}}_K})$ be the category of locally free $j^\dagger \mathcal{O}_{\overline{\mathcal{X}}_K}$ -modules E of finite rank endowed with an integrable connection of the form $\nabla : E \rightarrow E \otimes j_{\overline{\mathcal{X}}_K}^\dagger \Omega_{\overline{\mathcal{X}}_K}^1$. Then we have the equivalence of categories [1, 2.1.10, 2.2.3]

$$\text{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K) \xrightarrow{\sim} \text{MIC}(j^\dagger \mathcal{O}_{\overline{\mathcal{X}}_K}); \quad (V, (E, \nabla)) \mapsto j_V^\dagger (E, \nabla).$$

When $\mathcal{X}_K = \overline{\mathcal{X}}_K$, we denote the category $\text{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K)$ simply by $\text{MIC}(\mathcal{X}_K)$.

On the other hand, when we are given a pair (X, \overline{X}) , there exists a notion of overconvergent isocrystals on (X, \overline{X}) over K : Let us denote the category of overconvergent isocrystals on (X, \overline{X}) over K by $\text{Isoc}^\dagger(X, \overline{X})$. When $X = \overline{X}$, we call this category the category of convergent isocrystals on X over K and denote it by $\text{Isoc}(X)$.

Suppose that (X, \overline{X}) is a smooth pair which admits a lift $(X, \overline{X}) \hookrightarrow (\mathcal{X}, \overline{\mathcal{X}})$. Then we have the canonical fully faithful functor (called the functor of realization)

$$\Phi_{(\mathcal{X}, \overline{\mathcal{X}})} : \text{Isoc}^\dagger(X, \overline{X}) \longrightarrow \text{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K) = \text{MIC}(j^\dagger \mathcal{O}_{\overline{\mathcal{X}}_K}).$$

An object in $\text{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K) = \text{MIC}(j^\dagger \mathcal{O}_{\overline{\mathcal{X}}_K})$ is called overconvergent if it is in the essential image of $\Phi_{(\mathcal{X}, \overline{\mathcal{X}})}$. When $X = \overline{X}$ (hence $\mathcal{X} = \overline{\mathcal{X}}$), we denote the functor $\Phi_{(\mathcal{X}, \overline{\mathcal{X}})}$ by

$$\Phi_{\mathcal{X}} : \text{Isoc}(X) \longrightarrow \text{MIC}(\mathcal{X}_K),$$

and an object in $\text{MIC}(\mathcal{X}_K)$ is called convergent if it is in the essential image of $\Phi_{\mathcal{X}}$.

For a pair (X, \overline{X}) , the q -th power Frobenius map $F : (X, \overline{X}) \rightarrow (X, \overline{X})$ is a morphism over $\sigma^* : \text{Spf } \mathcal{O}_K \rightarrow \text{Spf } \mathcal{O}_K$ and so it induces a σ -linear functor $F^* : \text{Isoc}^\dagger(X, \overline{X}) \rightarrow \text{Isoc}^\dagger(X, \overline{X})$. An overconvergent F -isocrystal on (X, \overline{X}) over K is a pair (\mathcal{E}, Ψ) , where $\mathcal{E} \in \text{Isoc}(X, \overline{X})$ and Ψ is an isomorphism $F^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$. We denote the category of overconvergent F -isocrystals on (X, \overline{X}) over K by $F\text{-Isoc}^\dagger(X, \overline{X})$. When $X = \overline{X}$, we call this category the category of convergent F -isocrystals on X over K and denote it by $F\text{-Isoc}(X)$.

Suppose that (X, \overline{X}) is a smooth pair which admits a lift $(X, \overline{X}) \hookrightarrow (\mathcal{X}, \overline{\mathcal{X}})$ and an endomorphism $F : \overline{\mathcal{X}} \rightarrow \overline{\mathcal{X}}$ over σ^* lifting the q -th power Frobenius on

$\overline{\mathcal{X}}$. Then F induces an endomorphism on the formal smooth pair $(\mathcal{X}, \overline{\mathcal{X}})$ and an endomorphism on \mathcal{X}_K and on $\overline{\mathcal{X}}_K$, which we denote also by F . Let $F\text{-MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K)$ be the category of pairs $((V, (E, \nabla)), \Psi)$, where $(V, (E, \nabla)) \in \text{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K)$ and Ψ is an isomorphism $(F^{-1}(V), (F|_{F^{-1}(V)})^*(E, \nabla)) \xrightarrow{=} (V, (E, \nabla))$ in $\text{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K)$. Also, F induces the homomorphism $F^*j^\dagger\mathcal{O}_{\overline{\mathcal{X}}_K} \longrightarrow j^\dagger\mathcal{O}_{\overline{\mathcal{X}}_K}$ and so we can define the category $F\text{-MIC}(j^\dagger\mathcal{O}_{\overline{\mathcal{X}}_K})$ as the category of pairs $((E, \nabla), \Psi)$ consisting of $(E, \nabla) \in \text{MIC}(j^\dagger\mathcal{O}_{\overline{\mathcal{X}}_K})$ and an isomorphism $\Psi : F^*(E, \nabla) \xrightarrow{=} (E, \nabla)$. Then we have an equivalence of categories

$$F\text{-MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K) \xrightarrow{=} F\text{-MIC}(j^\dagger\mathcal{O}_{\overline{\mathcal{X}}_K}); \quad ((V, (E, \nabla)), \Psi) \mapsto (j_V^\dagger(E, \nabla), j_{V''}^\dagger\Psi)$$

(where V'' is a strict neighborhood on which Ψ is defined), and $\Phi_{(\mathcal{X}, \overline{\mathcal{X}})}$ induces the fully faithful functor

$$\Phi_{(\mathcal{X}, \overline{\mathcal{X}})} : F\text{-Isoc}^\dagger(X, \overline{X}) \longrightarrow F\text{-MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K) = F\text{-MIC}(j^\dagger\mathcal{O}_{\overline{\mathcal{X}}_K}).$$

When $X = \overline{X}$ (hence $\mathcal{X} = \overline{\mathcal{X}}$), we denote it by

$$\Phi_{\mathcal{X}} : F\text{-Isoc}(X) \longrightarrow F\text{-MIC}(\mathcal{X}_K).$$

Finally, we prepare a notation to express p -adic polyannulus: For a closed interval $[a, b]$ with $a, b \in [0, 1] \cap \Gamma^*$, we define the polyannulus $A_K^n[a, b]$ by

$$A_K^n[a, b] := \{x \in (\widehat{\mathbb{A}}_{O_K}^n)_K \mid \forall i, |t_i(x)| \in [a, b]\}$$

(where t_i is the i -th coordinate of $(\widehat{\mathbb{A}}_{O_K}^n)_K$).

Remark 1.1. We explained several notions only in the restricted cases: For example, the functor of realization can be defined for more general situation. For details on overconvergent isocrystals and ∇ -modules, see [1] and [11, §2], for example.

2 Generic convergence and overconvergence

In this section, we prove the following theorem, which claims that ‘generic convergence implies overconvergence’ in certain situation.

Theorem 2.1. *Let $(\mathcal{X}, \overline{\mathcal{X}})$ be a formal smooth pair. Let \mathcal{U} be an open dense p -adic formal subscheme of \mathcal{X} , denote the canonical morphism of formal smooth pairs $(\mathcal{U}, \mathcal{U}) \hookrightarrow (\mathcal{X}, \overline{\mathcal{X}})$ by $\alpha_{\mathcal{U}}$ and denote the induced functor $\text{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K) \longrightarrow \text{MIC}(\mathcal{U}_K)$ by $\alpha_{\mathcal{U}}^*$. Then an object (E, ∇) in $\text{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K)$ is overconvergent if and only if $\alpha_{\mathcal{U}}^*(E, \nabla)$ is convergent.*

Proof. The proof is divided into several steps.

Step 1: First we prove the ‘only if’ part. Let (X, \overline{X}) be the special fiber of $(\mathcal{X}, \overline{\mathcal{X}})$ and let U be the special fiber of \mathcal{U} . Then the ‘only if’ part follows from the following commutative diagram, which is the functoriality of the functors of the form $\Phi_{(?), (?)}$:

$$\begin{array}{ccc} \mathrm{Isoc}^\dagger(X, \overline{X}) & \xrightarrow{\Phi_{(\mathcal{X}, \overline{\mathcal{X}})}} & \mathrm{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K) \\ \alpha_{\mathcal{U}}^* \downarrow & & \alpha_{\mathcal{U}}^* \downarrow \\ \mathrm{Isoc}(U) & \xrightarrow{\Phi_{\mathcal{U}}} & \mathrm{MIC}(\mathcal{U}_K). \end{array}$$

(Here the functor $\alpha_{\mathcal{U}}^*$ on the right is the pull-back functor induced by the morphisms $\mathcal{U}_K \hookrightarrow \mathcal{X}_K, \mathcal{U}_K \hookrightarrow \overline{\mathcal{X}}_K$ induced by $\alpha_{\mathcal{U}}$.)

Step 2: Here we reduce the proof of the theorem to the case $\mathcal{U} = \mathcal{X}$. Let us factorize $\alpha_{\mathcal{U}}$ as

$$(\mathcal{U}, \mathcal{U}) \longrightarrow (\mathcal{X}, \mathcal{X}) \xrightarrow{\alpha_{\mathcal{X}}} (\mathcal{X}, \overline{\mathcal{X}}).$$

Then $\alpha_{\mathcal{X}}^*(E, \nabla)$ is an object in $\mathrm{MIC}(\mathcal{X}_K/K)$ whose restriction to $\mathrm{MIC}(\mathcal{U}_K/K)$ is convergent. So, by [16, 2.16], $\alpha_{\mathcal{X}}^*(E, \nabla)$ is convergent. So, if we assume that the theorem is true in the case $\mathcal{U} = \mathcal{X}$, we can conclude that (E, ∇) is overconvergent. In the following, we will assume that $\mathcal{U} = \mathcal{X}$.

Step 3: In this step, we follow the argument in [17, 2.4] and prove that the theorem is true if it is true for formal smooth pairs of the form $(\mathcal{X}_0, \overline{\mathcal{X}}_0) := (\widehat{\mathbb{G}}_{m, O_K}^n \times \widehat{\mathbb{A}}_{O_K}^m, \widehat{\mathbb{A}}_{O_K}^{n+m})$ ($n, m \in \mathbb{N}$) (and $\mathcal{U} = \mathcal{X}_0$). First, by claim in the proof of [17, Proposition 2.4], we have the following: For any closed point x of $\overline{\mathcal{X}}$, there exist open immersions $\overline{\mathcal{U}}_x \hookrightarrow \overline{\mathcal{X}}_x \hookrightarrow \overline{\mathcal{X}}$ with $x \in \overline{\mathcal{U}}_x$, an open formal subscheme \mathcal{X}_x of $\overline{\mathcal{X}}_x$ and a diagram of formal smooth pairs

$$(\mathcal{X}, \overline{\mathcal{X}}) \xleftarrow{j} (\mathcal{X}_x, \overline{\mathcal{X}}_x) \xrightarrow{f} (\widehat{\mathbb{G}}_{m, O_K}^n \times \widehat{\mathbb{A}}_{O_K}^m, \widehat{\mathbb{A}}_{O_K}^{n+m})$$

(where j is induced by the open immersion $\overline{\mathcal{X}}_x \hookrightarrow \overline{\mathcal{X}}$) for some n, m such that f is a strict finite etale morphism and that the morphism $(\overline{\mathcal{U}}_x \cap \mathcal{X}_x, \overline{\mathcal{U}}_x) \longrightarrow (\overline{\mathcal{U}}_x \cap \mathcal{X}, \overline{\mathcal{U}}_x)$ induced by j is an isomorphism.

For any closed point x in $\overline{\mathcal{X}}$, let us choose open immersions $\overline{\mathcal{U}}_x \hookrightarrow \overline{\mathcal{X}}_x \hookrightarrow \overline{\mathcal{X}}$ with $x \in \overline{\mathcal{U}}_x$, an open formal subscheme \mathcal{X}_x of $\overline{\mathcal{X}}_x$ and morphisms j, f as above. Then, if (E, ∇) is an object in $\mathrm{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K)$ such that $\alpha_{\mathcal{X}}^*(E, \nabla)$ is convergent, the restriction of it to $\mathrm{MIC}(\mathcal{X}_{x,K})$ is convergent for any x . Then, if we succeed to prove that the restriction of (E, ∇) to $\mathrm{MIC}(\mathcal{X}_{x,K}, \overline{\mathcal{X}}_{x,K})$ is overconvergent, it implies the overconvergence of the restriction of (E, ∇) to $\mathrm{MIC}(\overline{\mathcal{U}}_{x,K} \cap \mathcal{X}_{x,K}, \overline{\mathcal{U}}_{x,K}) = \mathrm{MIC}(\overline{\mathcal{U}}_{x,K} \cap \mathcal{X}_K, \overline{\mathcal{U}}_{x,K})$ for all x and this implies the overconvergence of (E, ∇) because it is known [17, 2.3] that the overconvergence condition for an object in $\mathrm{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K)$ has local nature with respect to the Zariski topology on $\overline{\mathcal{X}}$. So, it suffices to prove the theorem for $(\mathcal{X}_x, \overline{\mathcal{X}}_x)$, that is, we may assume that there exists a strict finite etale morphism $f : (\mathcal{X}, \overline{\mathcal{X}}) \longrightarrow (\widehat{\mathbb{G}}_{m, O_K}^n \times \widehat{\mathbb{A}}_{O_K}^m, \widehat{\mathbb{A}}_{O_K}^{n+m}) =: (\mathcal{X}_0, \overline{\mathcal{X}}_0)$ to prove the proposition.

Let (X_0, \overline{X}_0) be the special fiber of $(\mathcal{X}_0, \overline{\mathcal{X}}_0)$. Then, by the argument in the proof of [17, 2.4] and [19, 5.1], we have the push-forward functors

$$\begin{aligned} f_* &: \text{Isoc}^\dagger(X, \overline{X}) \longrightarrow \text{Isoc}^\dagger(X_0, \overline{X}_0), \\ f_* &: \text{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K) \longrightarrow \text{MIC}(\mathcal{X}_{0,K}, \overline{\mathcal{X}}_{0,K}) \end{aligned}$$

which makes the diagram

$$(2.1) \quad \begin{array}{ccc} \text{Isoc}^\dagger(X, \overline{X}) & \xrightarrow{\Phi_{(\mathcal{X}, \overline{\mathcal{X}})}} & \text{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K) \\ f_* \downarrow & & f_* \downarrow \\ \text{Isoc}^\dagger(X_0, \overline{X}_0) & \xrightarrow{\Phi_{(\mathcal{X}_0, \overline{\mathcal{X}}_0)}} & \text{MIC}(\mathcal{X}_{0,K}, \overline{\mathcal{X}}_{0,K}) \end{array}$$

commutative. Moreover, any object (E, ∇) in $\text{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K)$ is a direct summand of $f^*f_*(E, \nabla)$. Note that we have similar functors also in the convergent case (because the convergent case is a special case of overconvergent case): We have the the push-forward functors

$$\begin{aligned} f_* &: \text{Isoc}(X) \longrightarrow \text{Isoc}(X_0), \\ f_* &: \text{MIC}(\mathcal{X}_K) \longrightarrow \text{MIC}(\mathcal{X}_{0,K}) \end{aligned}$$

which makes the diagram

$$(2.2) \quad \begin{array}{ccc} \text{Isoc}(X) & \xrightarrow{\Phi_{\mathcal{X}}} & \text{MIC}(\mathcal{X}_K) \\ f_* \downarrow & & f_* \downarrow \\ \text{Isoc}(X_0) & \xrightarrow{\Phi_{\mathcal{X}_0}} & \text{MIC}(\mathcal{X}_{0,K}) \end{array}$$

commutative. Moreover, the diagrams (2.1), (2.2) are compatible via restriction functors $\alpha_{\mathcal{X}_0}^*$ because of the functoriality of $\Phi_{(?, ?)}$ and f_* .

Now let us assume that the theorem is true for $(\mathcal{X}_0, \overline{\mathcal{X}}_0)$ (and $\mathcal{U} = \mathcal{X}_0$) and let us take an object (E, ∇) in $\text{MIC}(\mathcal{X}_K, \overline{\mathcal{X}}_K)$ such that $\alpha_{\mathcal{X}}^*(E, \nabla)$ is convergent. Then $f_*\alpha_{\mathcal{X}}^*(E, \nabla) = \alpha_{\mathcal{X}_0}^*f_*(E, \nabla)$ is convergent by the diagram (2.2) and since we have assumed the theorem for $(\mathcal{X}_0, \overline{\mathcal{X}}_0)$, this implies the overconvergence of $f_*(E, \nabla)$. Then $f^*f_*(E, \nabla)$ is overconvergent and so (E, ∇) is also overconvergent because it is a direct summand. (See the characterization of overconvergence in [17, 2.1].) Therefore we have shown that it suffices to prove the theorem for $(\mathcal{X}_0, \overline{\mathcal{X}}_0)$ (and $\mathcal{U} = \mathcal{X}_0$).

Step 4: Here we prove the theorem for $(\mathcal{X}_0, \overline{\mathcal{X}}_0)$ above, by using [17, 2.7]. Let (E, ∇) be an object in $\text{MIC}(\mathcal{X}_{0,K}, \overline{\mathcal{X}}_{0,K})$. In this case, (E, ∇) is defined on $A_K^n[\lambda, 1] \times A_K^m[0, 1]$ for some $\lambda \in [0, 1] \cap \Gamma^*$. So we have the notion of the intrinsic generic radius of convergence $IR(E, \rho)$ for $\rho \in [\lambda, 1]^n \times [0, 1]^m$ which is defined by Kedlaya-Xiao [14]. By [17, 2.7], (E, ∇) is overconvergent if and only if $IR(E, 1) = 1$, where

$\mathbf{1} := (1, \dots, 1)$. On the other hand, by applying [17, 2.1] (see also [11, 2.5.6–8], [1, 2.2.13]) to $(\mathcal{X}_{0,K}, \mathcal{X}_{0,K})$, we see the following: When we fix a set of generators $(e_\alpha)_\alpha$ of $\Gamma(\mathcal{X}_{0,K}, E)$, $\alpha_{\mathcal{X}_0}^*(E, \nabla)$ is convergent if and only if, for each $\eta \in (0, 1) \cap \Gamma^*$ and any α , the multi-sequence

$$\left\{ \left\| \frac{1}{i_1! \cdots i_{n+m}!} \partial_1^{i_1} \cdots \partial_{n+m}^{i_{n+m}}(e_\alpha) \right\| \eta^{i_1 + \cdots + i_{n+m}} \right\}_{i_1, \dots, i_{n+m}}$$

tends to zero as $i_1, \dots, i_{n+m} \rightarrow \infty$, where $\partial_j := \frac{\partial}{\partial t_j}$ and $\|\cdot\|$ denotes any p -adic Banach norm on $\Gamma(\mathcal{X}_{0,K}, E)$ induced by the affinoid norm on $\Gamma(\mathcal{X}_{0,K}, \mathcal{O})$. Then, by exactly the same argument as the proof of [17, 2.7], we see that the above condition is equivalent to the condition that $IR(E, \mathbf{1}) = \min_i \{p^{-1/(p-1)} |\partial_i|_{E, \mathbf{1}, \text{sp}}^{-1}\} > \eta$ for each $\eta \in (0, 1) \cap \Gamma^*$ (where $|\partial_i|_{E, \mathbf{1}, \text{sp}}$ is the spectral norm of ∂_i on E ‘at the generic point of radius $\mathbf{1}$ ’), and it is nothing but the condition $IR(E, \mathbf{1}) = 1$. Hence (E, ∇) is overconvergent if and only if $\alpha_{\mathcal{X}_0}^*(E, \nabla)$ is convergent and so we are done. \square

We have the following corollary for convergent F -isocrystals.

Corollary 2.2. *Let (X, \overline{X}) be a smooth pair, let U be an dense open subscheme of X and let \mathcal{E} be an object in $F\text{-Isoc}(U)$ satisfying the following condition $(*)$:*

$(*)$: *There exists a Zariski open covering $\overline{X} = \bigcup_{\alpha \in \Delta} \overline{X}_\alpha$, lifts $(X \cap \overline{X}_\alpha, \overline{X}_\alpha \hookrightarrow (\mathcal{X}_\alpha, \overline{\mathcal{X}}_\alpha))$ and endomorphisms $F : \overline{\mathcal{X}}_\alpha \rightarrow \overline{\mathcal{X}}_\alpha$ lifting the q -th power Frobenius on \overline{X}_α (for $\alpha \in \Delta$) satisfying the following: If we denote the open formal subscheme of $\overline{\mathcal{X}}_\alpha$ with special fiber $U \cap \overline{X}_\alpha$ by \mathcal{U}_α , the image of \mathcal{E} by the composite*

$$(2.3) \quad F\text{-Isoc}(U) \rightarrow F\text{-Isoc}(U \cap \overline{X}_\alpha) \xrightarrow{\Phi_{\mathcal{U}_\alpha}} F\text{-MIC}(\mathcal{U}_\alpha)$$

is in the essential image of the functor

$$(2.4) \quad F\text{-MIC}(\mathcal{X}_{\alpha,K}, \overline{\mathcal{X}}_{\alpha,K}) \rightarrow F\text{-MIC}(\mathcal{U}_\alpha).$$

Then \mathcal{E} is in the essential image of the functor $F\text{-Isoc}^\dagger(X, \overline{X}) \rightarrow F\text{-Isoc}(U)$.

Proof. Assume that \mathcal{E} is sent as $\mathcal{E} \mapsto \mathcal{E}_\alpha \mapsto E_\alpha$ by the functors in (2.3) and let \tilde{E}_α be an object in $F\text{-MIC}(\mathcal{X}_{\alpha,K}, \overline{\mathcal{X}}_{\alpha,K})$ which is sent to E_α by the functor (2.4). Then, by Theorem 2.1, \tilde{E}_α is overconvergent, that is, there exists an object $\tilde{\mathcal{E}}_\alpha \in F\text{-Isoc}^\dagger(X \cap \overline{X}_\alpha, \overline{X}_\alpha)$ which is sent to E_α by $\Phi_{(\mathcal{X}_\alpha, \overline{\mathcal{X}}_\alpha)}$. Now let us consider the following commutative diagram:

$$\begin{array}{ccc} F\text{-Isoc}^\dagger(X \cap \overline{X}_\alpha, \overline{X}_\alpha) & \xrightarrow{\Phi_{(\mathcal{X}_\alpha, \overline{\mathcal{X}}_\alpha)}} & F\text{-MIC}(\mathcal{X}_{\alpha,K}, \overline{\mathcal{X}}_{\alpha,K}) \\ \downarrow & & \downarrow \\ F\text{-Isoc}(U \cap \overline{X}_\alpha) & \xrightarrow{\Phi_{\mathcal{U}_\alpha}} & F\text{-MIC}(\mathcal{U}_\alpha). \end{array}$$

From the commutativity of the diagram, we see that both \mathcal{E}_α and the restriction of $\tilde{\mathcal{E}}_\alpha$ to $F\text{-Isoc}(U \cap \overline{X}_\alpha)$ is sent to E_α by $\Phi_{\mathcal{U}_\alpha}$. Since $\Phi_{\mathcal{U}_\alpha}$ is fully faithful, we can conclude that they are isomorphic.

Now let us glue the $\tilde{\mathcal{E}}_\alpha$'s: Since \mathcal{E}_α 's ($\alpha \in \Delta$) are the restriction of \mathcal{E} , they form a descent data on $F\text{-Isoc}(U \cap \overline{X}_\alpha \cap \overline{X}_{\alpha'})$ and $F\text{-Isoc}(U \cap \overline{X}_\alpha \cap \overline{X}_{\alpha'} \cap \overline{X}_{\alpha''})$ ($\alpha, \alpha', \alpha'' \in \Delta$). Then, since the restriction functor

$$F\text{-Isoc}^\dagger(X \cap \overline{X}_\alpha, \overline{X}_\alpha) \longrightarrow F\text{-Isoc}(U \cap \overline{X}_\alpha)$$

(and the functor we obtain by replacing \overline{X}_α by $\overline{X}_\alpha \cap \overline{X}_{\alpha'}$ or $\overline{X}_\alpha \cap \overline{X}_{\alpha'} \cap \overline{X}_{\alpha''}$) is fully faithful by [19, 4.1.1] and [12, 4.2.1] (see also [9]), we see that $\tilde{\mathcal{E}}_\alpha$'s also form a descent data on $F\text{-Isoc}^\dagger(X \cap \overline{X}_\alpha \cap \overline{X}_{\alpha'}, \overline{X}_\alpha \cap \overline{X}_{\alpha'})$ and $F\text{-Isoc}^\dagger(X \cap \overline{X}_\alpha \cap \overline{X}_{\alpha'} \cap \overline{X}_{\alpha''}, \overline{X}_\alpha \cap \overline{X}_{\alpha'} \cap \overline{X}_{\alpha''})$ ($\alpha, \alpha', \alpha'' \in \Delta$). Hence $\tilde{\mathcal{E}}_\alpha$'s glue to an object $\tilde{\mathcal{E}}$ in $F\text{-Isoc}^\dagger(X, \overline{X})$ whose restriction to $F\text{-Isoc}(U)$ is \mathcal{E} . So we are done. \square

3 Purity for overconvergence

In this section, we will prove the following theorem, which is the main result in this paper.

Theorem 3.1. *Let (X, \overline{X}) be a smooth pair and let $\overline{Z} \subseteq Z \subseteq \overline{X}$ be closed subschemes of codimension at least 2. Then the restriction functor*

$$(3.1) \quad F\text{-Isoc}^\dagger(X, \overline{X}) \longrightarrow F\text{-Isoc}^\dagger(X \setminus Z, \overline{X} \setminus \overline{Z})$$

is an equivalence of categories.

Note that, in the case where \overline{Z} is empty, this is reduced to the purity theorem of Kedlaya [11, 5.3.3].

Before the proof, note that the functor (3.1) is fully faithful because we have the fully faithful functors

$$F\text{-Isoc}^\dagger(X, \overline{X}) \longrightarrow F\text{-Isoc}^\dagger(X \setminus Z, \overline{X}) \longrightarrow F\text{-Isoc}(X \setminus Z),$$

$$F\text{-Isoc}^\dagger(X \setminus Z, \overline{X} \setminus \overline{Z}) \longrightarrow F\text{-Isoc}(X \setminus Z).$$

(The full faithfulness of the first functor in the first line is proven in [19, 4.1.1] and the full faithfulness for other functors are proven in [12, 4.2.1].) So it suffices to prove that the functor (3.1) is essentially surjective.

Proof. The proof is divided into several steps.

Step 1: First note that it suffices to prove the theorem in the case $Z = \overline{Z}$. Indeed, we can factorize the functor (3.1) as

$$F\text{-Isoc}^\dagger(X, \overline{X}) \longrightarrow F\text{-Isoc}^\dagger(X \setminus \overline{Z}, \overline{X} \setminus \overline{Z}) \longrightarrow F\text{-Isoc}^\dagger(X \setminus Z, \overline{X} \setminus \overline{Z})$$

and the second functor is an equivalence of categories by [11, 5.3.3]. So it suffices to prove the equivalence of the first functor. In the following, we always assume that $Z = \overline{Z}$.

Step 2: Next, note that it suffices to prove the theorem in the following cases:

- (a) The case $Z \subseteq \overline{X} \setminus X$.
- (b) The case $Z \subseteq X$.

Indeed, if we put $Z' := Z \cap (\overline{X} \setminus X)$, we can factorize the functor (3.1) (with $Z = \overline{Z}$) as

$$F\text{-Isoc}^\dagger(X, \overline{X}) \longrightarrow F\text{-Isoc}^\dagger(X, \overline{X} \setminus Z') \longrightarrow F\text{-Isoc}^\dagger(X \setminus Z, \overline{X} \setminus Z)$$

and the equivalence of the first (resp. the second) functor follows from the theorem in the case (a) (resp. (b)).

Step 3: Let us prove that we may assume that Z is smooth. For $n \in \mathbb{N}$, let us denote the q^n -th power map $\text{Spec } k \longrightarrow \text{Spec } k$ simply by q^n and for a k -scheme S , let us put $S^{(n)} := S \times_{\text{Spec } k, q^n} \text{Spec } k$. Then the commutative diagram

$$\begin{array}{ccccc} (X, \overline{X}) & \xrightarrow{F_{\text{rel}}} & (X^{(n)}, \overline{X}^{(n)}) & \longrightarrow & (X, \overline{X}) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec } k & \xlongequal{\quad} & \text{Spec } k & \xrightarrow{q^n} & \text{Spec } k \\ \cap \downarrow & & \cap \downarrow & & \cap \downarrow \\ \text{Spf } O_K & \xlongequal{\quad} & \text{Spf } O_K & \xrightarrow{\sigma^{*n}} & \text{Spf } O_K \end{array}$$

(where the upper left square is Cartesian and F_{rel} is the relative Frobenius morphism associated to the q^n -th power maps) induces the functors

$$(3.2) \quad F\text{-Isoc}^\dagger(X, \overline{X}) \xrightarrow{\alpha} F\text{-Isoc}^\dagger(X^{(n)}, \overline{X}^{(n)}) \xrightarrow{\beta} F\text{-Isoc}^\dagger(X, \overline{X}).$$

Then, for $(\mathcal{E}, \Psi) \in F\text{-Isoc}^\dagger(X, \overline{X})$, we have

$$\beta \circ \alpha(\mathcal{E}, \Psi) = (F^{*n}\mathcal{E}, F^{*n}\Psi) \xrightarrow{\cong} (\mathcal{E}, \Psi),$$

where the last isomorphism is the composite

$$(F^{*n}\mathcal{E}, F^{*n}\Psi) \xrightarrow{F^{*n-1}\Psi} (F^{*n-1}\mathcal{E}, F^{*n}\Psi) \xrightarrow{F^{*n-1}\Psi} \dots \xrightarrow{\Psi} (\mathcal{E}, \Psi).$$

Hence β is essentially surjective. By the same argument, we see that the functors

$$F\text{-Isoc}^\dagger(X^{(n)}, \overline{X}^{(n)} \setminus Z^{(n)}) \longrightarrow F\text{-Isoc}^\dagger(X, \overline{X} \setminus Z), \quad (\text{in the case (a)})$$

$$F\text{-Isoc}^\dagger(X^{(n)} \setminus Z^{(n)}, \overline{X}^{(n)} \setminus Z^{(n)}) \longrightarrow F\text{-Isoc}^\dagger(X \setminus Z, \overline{X} \setminus Z) \quad (\text{in the case (b)})$$

are also essentially surjective. So, to prove the theorem for the pair (X, \overline{X}) and the closed subscheme Z in the case (a) or (b), it suffices to prove the theorem for the pair $(X^{(n)}, \overline{X}^{(n)})$ and the closed subscheme $Z^{(n)}$ for some n , and then it suffices to prove the theorem for the pair $(X^{(n)}, \overline{X}^{(n)})$ and the closed subscheme $Z_{\text{red}}^{(n)}$ (= the reduced closed subscheme of $Z^{(n)}$ with same underlying topological space).

Now let us take $n \in \mathbb{N}$ in order that $Z_{\text{red}}^{(n)}$ is generically smooth. Let $Z_0 \subseteq Z_{\text{red}}^{(n)}$ be a dense open subscheme such that Z_0 is smooth and put $Z_1 := Z_{\text{red}}^{(n)} \setminus Z_0$. Then in the case (a), we can factorize the functor

$$F\text{-Isoc}^\dagger(X^{(n)}, \overline{X}^{(n)}) \longrightarrow F\text{-Isoc}^\dagger(X^{(n)}, \overline{X}^{(n)} \setminus Z_{\text{red}}^{(n)})$$

(the functor (3.1) for $(X^{(n)}, \overline{X}^{(n)})$ and $Z_{\text{red}}^{(n)}$) as

$$(3.3) \quad F\text{-Isoc}^\dagger(X^{(n)}, \overline{X}^{(n)}) \rightarrow F\text{-Isoc}^\dagger(X^{(n)}, \overline{X}^{(n)} \setminus Z_1) \rightarrow F\text{-Isoc}^\dagger(X^{(n)}, \overline{X}^{(n)} \setminus Z_{\text{red}}^{(n)})$$

and $(\overline{X}^{(n)} \setminus Z_1) \setminus (\overline{X}^{(n)} \setminus Z_{\text{red}}^{(n)}) = Z_0$ is smooth. So, if we assume the theorem in the case Z is smooth, the second functor in (3.3) is an equivalence. Moreover, the dimension of $\overline{X}^{(n)}$ is equal to the dimension of \overline{X} and the codimension of Z_1 in $\overline{X}^{(n)}$ is strictly greater than that of Z in \overline{X} . So we can prove the equivalence of the first functor in (3.3) by descending induction on the codimension of Z in \overline{X} . We can prove the case (b) in the same way.

Step 4: Note that we can work Zariski locally on \overline{X} : Indeed, the categories $F\text{-Isoc}^\dagger(X, \overline{X})$, $F\text{-Isoc}^\dagger(X \setminus Z, \overline{X} \setminus Z)$ satisfy the descent property with respect to the Zariski topology on \overline{X} and we already know the full faithfulness of the functor (3.1) (or its analogue when we shrink \overline{X}), the descent data concerning $F\text{-Isoc}^\dagger(X \setminus Z, \overline{X} \setminus Z)$ lifts to the descent data concerning $F\text{-Isoc}^\dagger(X, \overline{X})$. So it suffices to prove the theorem Zariski locally on \overline{X} .

Step 5: Here we prove that it suffices to prove the theorem in the case (a) with Z smooth. To prove this, it suffices to reduce the case (b) with Z smooth to the case (a) with Z smooth. So assume that we are in the case (b) with Z smooth. Then, since Z does not meet $\overline{X} \setminus X$ and we can work Zariski locally on \overline{X} , we may assume that either Z or $\overline{X} \setminus X$ is empty. In the case Z is empty, the theorem is obvious. In the case $\overline{X} \setminus X$ is empty, it suffices to prove the equivalence of the functor $F\text{-Isoc}^\dagger(X, X) \longrightarrow F\text{-Isoc}^\dagger(X \setminus Z, X \setminus Z)$. This functor is factorized as

$$(3.4) \quad F\text{-Isoc}^\dagger(X, X) \longrightarrow F\text{-Isoc}^\dagger(X \setminus Z, X) \longrightarrow F\text{-Isoc}^\dagger(X \setminus Z, X \setminus Z)$$

and the first functor is an equivalence by [11, 5.3.3]. So it suffices to prove that the second functor is an equivalence. Since we can consider Zariski locally and since Z is smooth, we may assume that there exists a smooth divisor $D \subseteq X$ containing Z . Then, by [11, 5.3.7] we have the canonical equivalence of categories

$$F\text{-Isoc}^\dagger(X \setminus Z, X) \xrightarrow{=} F\text{-Isoc}^\dagger(X \setminus Z, X \setminus Z) \times_{F\text{-Isoc}^\dagger(X \setminus D, X \setminus Z)} F\text{-Isoc}^\dagger(X \setminus D, X)$$

via which the second functor in (3.4) is regarded as the first projection

$$F\text{-Isoc}^\dagger(X \setminus Z, X \setminus Z) \times_{F\text{-Isoc}^\dagger(X \setminus D, X \setminus Z)} F\text{-Isoc}^\dagger(X \setminus D, X) \longrightarrow F\text{-Isoc}^\dagger(X \setminus Z, X \setminus Z).$$

So it suffices to prove the equivalence of the functor $F\text{-Isoc}^\dagger(X \setminus D, X) \longrightarrow F\text{-Isoc}^\dagger(X \setminus D, X \setminus Z)$ to prove the theorem in this case, and it is nothing but the theorem in the case (a). it suffices to prove the theorem in the case (a) with Z smooth.

Step 6: Let us put $Y := \overline{X} \setminus X$ (which is a simple normal crossing divisor in \overline{X}) and let $Y = \bigcup_{i=1}^r Y_i$ be the decomposition of Y into the irreducible components (so r denotes the number of irreducible components of Y). For a subset $I \subseteq [1, r]$, we put $Y_I := \bigcap_{i \in I} Y_i$ and for $s \in \mathbb{N}$, we put $Y^{(s)} := \bigcup_{|I|=s} Y_I$. In this step, we show that it suffices to prove the theorem in the following cases:

(a-1) : The case (a) with $r = 1$ and Z smooth.

(a-s) ($s \geq 2$): The case (a) with $r = s$, $Y_{[1,s]} \neq \emptyset$ and $Z = Y^{(2)}$.

(Note that, in the cases (a-s) ($s \geq 2$), Z is no more smooth.)

Let us assume that the theorem is true in the case (a-s) ($s \geq 1$) and let $(X, \overline{X}), Z \subseteq \overline{X}$ be as in case (a) with Z smooth. First, let us note that, to prove the theorem for this (X, \overline{X}) and $Z \subseteq \overline{X}$, it suffices to prove the equivalence of the composite functor

$$(3.5) \quad F\text{-Isoc}^\dagger(X, \overline{X}) \longrightarrow F\text{-Isoc}^\dagger(X, \overline{X} \setminus Z) \longrightarrow F\text{-Isoc}^\dagger(X, \overline{X} \setminus (Y^{(2)} \cup Z)),$$

because they are fully faithful. We can factor the functor (3.5) as follows:

$$F\text{-Isoc}^\dagger(X, \overline{X}) \longrightarrow F\text{-Isoc}^\dagger(X, \overline{X} \setminus Y^{(2)}) \longrightarrow F\text{-Isoc}^\dagger(X, \overline{X} \setminus (Y^{(2)} \cup Z)).$$

So, to prove the theorem, it suffices to prove the equivalence of the functors

$$(3.6) \quad F\text{-Isoc}^\dagger(X, \overline{X} \setminus Y^{(2)}) \longrightarrow F\text{-Isoc}^\dagger(X, \overline{X} \setminus (Y^{(2)} \cup Z)),$$

$$(3.7) \quad F\text{-Isoc}^\dagger(X, \overline{X}) \longrightarrow F\text{-Isoc}^\dagger(X, \overline{X} \setminus Y^{(2)}).$$

The equivalence of the functor (3.6) is nothing but the theorem for the pair $(X, \overline{X} \setminus Y^{(2)})$ and the closed subscheme $Z \setminus Y^{(2)} \subseteq \overline{X} \setminus Y^{(2)}$. Note that $Y \setminus Y^{(2)} = (\overline{X} \setminus Y^{(2)}) \setminus X$ is a smooth divisor and $Z \setminus Y^{(2)}$ is smooth. Hence we are locally in the situation in the case (a) with $r = 1$, Z smooth, that is, the situation in the case (a-1). On the other hand, we can reduce the equivalence of the functor (3.7) to (a-s) ($s \geq 2$) in the following way: For any closed point $x \in \overline{X}$, let $Y_{\ni x} := \bigcup_{i \in I_x} Y_i$ be the union of irreducible components of Y containing x . Then we can take an open subscheme \overline{X}_x of \overline{X} containing x such that $Y \cap \overline{X}_x = Y_{\ni x} \cap \overline{X}_x$. Since we can consider Zariski locally, the equivalence of the functor (3.7) is reduced to the theorem for the pair $(X \cap \overline{X}_x, \overline{X}_x)$ and the closed subscheme $(Y \cap \overline{X}_x)^{(2)} = Y_{\ni x}^{(2)} \cap \overline{X}_x$. Let us put $s := |I_x|$. Then, when $s \geq 2$, it is in the situation (a-s). When $s \leq 1$, it is trivially true since $Y_{\ni x}^{(2)} = \emptyset$ in this case. Therefore, we are reduced to the case (a-s) ($s \geq 1$).

Step 7: In this step, we show that it suffices to prove the theorem in the case (a- s) with $s \geq 2$. To do this, it suffices to reduce the proof of the theorem in the case (a-1) to the case (a-2). So assume that we are in the situation (a-1). Then, since we may consider locally, we may assume that there exists a smooth divisor $Y' \subseteq \overline{X}$ which meets Y transversally such that $Y' \cap Y$ contains Z . In this case, it suffices to prove the equivalence of the composite functor

$$F\text{-Isoc}(X, \overline{X}) \longrightarrow F\text{-Isoc}(X, \overline{X} \setminus Z) \longrightarrow F\text{-Isoc}(X, \overline{X} \setminus (Y \cap Y')).$$

Then, by [11, 5.3.7], we can rewrite the above composite as follows:

$$\begin{aligned} F\text{-Isoc}^\dagger(X, \overline{X}) &\xrightarrow{=} F\text{-Isoc}^\dagger(X \setminus Y', \overline{X}) \times_{F\text{-Isoc}^\dagger(X \setminus Y', X)} F\text{-Isoc}^\dagger(X, X) \\ &\longrightarrow F\text{-Isoc}^\dagger(X \setminus Y', \overline{X} \setminus (Y \cap Y')) \times_{F\text{-Isoc}^\dagger(X \setminus Y', X)} F\text{-Isoc}^\dagger(X, X) \\ &\xleftarrow{=} F\text{-Isoc}^\dagger(X, \overline{X} \setminus (Y \cap Y')). \end{aligned}$$

So the equivalence we need is reduced to the equivalence of the functor

$$F\text{-Isoc}^\dagger(X \setminus Y', \overline{X}) \longrightarrow F\text{-Isoc}^\dagger(X \setminus Y', \overline{X} \setminus (Y \cap Y'))$$

and this is the theorem for the pair $(X \setminus Y', \overline{X})$ and the closed subscheme $Y \cap Y' \subseteq \overline{X}$. Since this is contained in the case (a-2), we are done.

Step 8: Assume now that we are in the situation (a- s) with $s \geq 2$. Let us take a closed point $x \in \overline{X}$. Let $Y_{\ni x} = \bigcup_{i=1}^{s'} Y_i$ be the union of irreducible components of Y containing x ($s' \leq s$) and let $Y_{\not\ni x}$ be the union of irreducible components of Y not containing x . By applying [10, Theorem 2] to $\overline{X} \setminus Y_{\not\ni x}$ and the simple normal crossing divisor $Y \setminus Y_{\not\ni x} = Y_{\ni x} \setminus Y_{\not\ni x}$ on it, we see that there exists an open subscheme \overline{X}_x in $\overline{X} - Y_{\not\ni x}$ containing x and a finite etale morphism $f_0 : \overline{X}_x \longrightarrow \mathbb{A}_k^d$ for some $d \geq s'$ such that, for $1 \leq i \leq s'$, $f_0(Y_i \cap \overline{X}_x)$ is contained in the i -th coordinate hyperplane H_i of \mathbb{A}_k^d . Then $Y_i \cap \overline{X}_x \subseteq f_0^{-1}(H_i)$ is an open and closed immersion, and so $Y \cap \overline{X}_x = Y_{\ni x} \cap \overline{X}_x$ is a simple normal crossing subdivisor of $\bigcup_{i=1}^n f_0^{-1}(H_i)$.

Since we can consider Zariski locally, it suffices to prove the theorem for the pair $(X \cap \overline{X}_x, \overline{X}_x)$ and the closed subscheme $Z \cap \overline{X}_x = Y^{(2)} \cap \overline{X}_x = Y_{\ni x}^{(2)} \cap \overline{X}_x$. It is contained in the case (a- s') when $s \geq 2$. When $s' \leq 1$, the theorem in this case is trivially true since $Y_{\ni x}^{(2)} = \emptyset$. Summing up the argument here, we see the following: It suffices to prove the theorem in the situation (a- s) with $s \geq 2$ which admits a finite etale morphism $f : \overline{X} \longrightarrow \mathbb{A}_k^d$ (for some $d \geq s$) such that $Y = \overline{X} \setminus X$ is a simple normal crossing subdivisor of $\tilde{Y} := \bigcup_{i=1}^s f^{-1}(H_i)$, where H_i denotes the i -th coordinate hyperplane of \mathbb{A}_k^d .

Step 9: Let the notation be as above and let us put $X' := X \setminus \tilde{Y}$, $\tilde{Z} := \tilde{Y}^{(2)}$. Consider the following commutative diagram:

$$(3.8) \quad \begin{array}{ccccc} F\text{-Isoc}^\dagger(X, \overline{X}) & \longrightarrow & F\text{-Isoc}^\dagger(X', \overline{X}) & \longrightarrow & F\text{-Isoc}^\dagger(X', \overline{X} \setminus \tilde{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ F\text{-Isoc}^\dagger(X, X) & \longrightarrow & F\text{-Isoc}^\dagger(X', X) & \longrightarrow & F\text{-Isoc}^\dagger(X', X \setminus \tilde{Z}). \end{array}$$

Now assume that the theorem is true for the pair (X', \overline{X}) and the closed subscheme $\tilde{Z} \subseteq \overline{X}$, and let us take an object \mathcal{E} in $F\text{-Isoc}^\dagger(X, \overline{X} \setminus Z)$. Then the restriction of \mathcal{E} to $F\text{-Isoc}^\dagger(X', \overline{X} \setminus \tilde{Z})$ extends to an object \mathcal{F} in $F\text{-Isoc}^\dagger(X', \overline{X})$. The restriction of \mathcal{F} to $F\text{-Isoc}^\dagger(X', X)$ is canonically isomorphic to the restriction of \mathcal{E} because so do they in the category $F\text{-Isoc}^\dagger(X', X \setminus \tilde{Z})$ (note that all the functors in (3.8) are fully faithful). Now let us note that the left square is Cartesian in the sense that the induced functor

$$(3.9) \quad F\text{-Isoc}^\dagger(X, \overline{X}) \longrightarrow F\text{-Isoc}^\dagger(X', \overline{X}) \times_{F\text{-Isoc}^\dagger(X', X)} F\text{-Isoc}^\dagger(X, X)$$

is an equivalence of categories ([11, 5.3.7]). Hence the object $(\mathcal{F}, \text{the restriction of } \mathcal{E})$ in the target of (3.9) lifts to an object $\tilde{\mathcal{E}}$ in $F\text{-Isoc}^\dagger(X, \overline{X})$ and it gives the lift of \mathcal{E} . So we see that it suffices to prove the theorem for the pair (X', \overline{X}) and the closed subscheme $\tilde{Z} \subseteq \overline{X}$: That is, to prove the theorem, we may assume that there exists a finite etale morphism $f : \overline{X} \longrightarrow \mathbb{A}_k^d$ and some $s \leq d$ such that $Y := \overline{X} \setminus X = \bigcup_{i=1}^s f^{-1}(H_i)$ and that $Z = Y^{(2)}$. (Note that we do not assume the condition (a-s) any more.)

Step 10: Let the notation be as above that let us put $\overline{X}_0 := \mathbb{A}_k^d, X_0 := \mathbb{A}_k^d \setminus \bigcup_{i=1}^s H_i, Y_0 := \bigcup_{i=1}^s H_i, Z_0 := Y_0^{(2)}$. Then, by [19, 5.1], we have the puch-out functors f_* as in the following commutative diagram:

$$(3.10) \quad \begin{array}{ccc} F\text{-Isoc}^\dagger(X, \overline{X}) & \longrightarrow & F\text{-Isoc}^\dagger(X, \overline{X} \setminus Z) \\ f_* \downarrow & & f_* \downarrow \\ F\text{-Isoc}^\dagger(X_0, \overline{X}_0) & \longrightarrow & F\text{-Isoc}^\dagger(X_0, \overline{X}_0 \setminus Z_0). \end{array}$$

Moreover, it is known that, for any $\mathcal{E} \in F\text{-Isoc}^\dagger(X, \overline{X} \setminus Z)$, there exist morphisms $\mathcal{E} \xrightarrow{\alpha} f^* f_* \mathcal{E} \xrightarrow{\beta} \mathcal{E}$ which makes \mathcal{E} a direct summand of $f^* f_* \mathcal{E}$. (See [11, 2.6.8].) Now let us assume that the theorem is true for the pair (X_0, \overline{X}_0) and the closed subscheme $Z_0 \subseteq \overline{X}_0$, and let us take an object \mathcal{E} in $F\text{-Isoc}^\dagger(X, \overline{X} \setminus Z)$. Then there exists an object \mathcal{F} in $F\text{-Isoc}^\dagger(X_0, \overline{X}_0)$ which restricts to $f_* \mathcal{E}$ in $F\text{-Isoc}^\dagger(X_0, \overline{X}_0 \setminus Z_0)$. Then $f^* \mathcal{F} \in F\text{-Isoc}^\dagger(X, \overline{X})$ restricts to $f^* f_* \mathcal{E}$ in $F\text{-Isoc}^\dagger(X, \overline{X} \setminus Z)$. Now let us note that \mathcal{E} is isomorphic to the image of the composite $\alpha \circ \beta : f^* f_* \mathcal{E} \longrightarrow f^* f_* \mathcal{E}$. Since the upper horizontal functor in (3.10) is fully faithful and exact, $\alpha \circ \beta$ lifts to an endomorphism $\gamma : f^* \mathcal{F} \longrightarrow f^* \mathcal{F}$ of $f^* \mathcal{F}$ and $\text{Im } \gamma$ is an object in $F\text{-Isoc}^\dagger(X, \overline{X})$ which restricts to \mathcal{E} in $F\text{-Isoc}^\dagger(X, \overline{X} \setminus Z)$. So the theorem for (X, \overline{X}) and $Z \subseteq \overline{X}$ is true. Hence we have proved that it suffices to prove the theorem for the pair (X_0, \overline{X}_0) and the closed subscheme $Z_0 \subseteq \overline{X}_0$.

Step 11: Let (X_0, \overline{X}_0) and $Z_0 \subseteq \overline{X}_0$ be as above. In this step, we finish the proof of the theorem by giving a proof of the theorem for (X_0, \overline{X}_0) and $Z_0 \subseteq \overline{X}_0$.

Let us put $\overline{\mathcal{X}}_0 := \widehat{\mathbb{A}}_{O_K}^d, \mathcal{X}_0 := \widehat{\mathbb{G}}_{m, O_K}^s \times \widehat{\mathbb{A}}_{O_K}^{d-s}$. Let t_1, \dots, t_d be the coordinate function of $\overline{\mathcal{X}}_0$ and let $F : \overline{\mathcal{X}}_0 \longrightarrow \overline{\mathcal{X}}_0$ be the morphism over $\sigma^* : \text{Spf } O_K \longrightarrow \text{Spf } O_K$ defined by $F^*(t_i) := t_i^q$ ($1 \leq i \leq d$). Then $(\mathcal{X}_0, \overline{\mathcal{X}}_0)$ is a formal smooth pair with

special fiber (X_0, \overline{X}_0) and F defines an endomorphism on it which lifts the q -th power Frobenius on \overline{X}_0 . Let $\overline{\mathcal{X}}'_0$ be the open formal subscheme of $\overline{\mathcal{X}}_0$ with special fiber $\overline{X}_0 \setminus Z_0$. Then we have the commutative diagram

$$\begin{array}{ccc} F\text{-Isoc}^\dagger(X_0, \overline{X}_0 \setminus Z_0) & \xrightarrow{\Phi_{(\mathcal{X}_0, \overline{\mathcal{X}}'_0)}} & F\text{-MIC}(\mathcal{X}_{0,K}, \overline{\mathcal{X}}'_{0,K}) \\ \downarrow & & \downarrow \\ F\text{-Isoc}(X_0) & \xrightarrow{\Phi_{\mathcal{X}_0}} & F\text{-MIC}(\mathcal{X}_{0,K}). \end{array}$$

Let \mathcal{E} be an object in $F\text{-Isoc}^\dagger(X_0, \overline{X}_0 \setminus Z_0)$ and put $E := \Phi_{(\mathcal{X}_0, \overline{\mathcal{X}}'_0)}(\mathcal{E})$. To prove the theorem, it suffices to find an object in $F\text{-Isoc}^\dagger(X_0, \overline{X}_0)$ whose restriction to $F\text{-Isoc}(X_0)$ is isomorphic to the restriction of \mathcal{E} to $F\text{-Isoc}(X_0)$, because the functor $F\text{-Isoc}^\dagger(X_0, \overline{X}_0 \setminus Z_0) \rightarrow F\text{-Isoc}(X_0)$ is fully faithful. Then, by Corollary 2.2, it suffices to prove that the restriction of E to $F\text{-MIC}(\mathcal{X}_{0,K})$ is extendable to an object in $F\text{-MIC}(\mathcal{X}_{0,K}, \overline{\mathcal{X}}_{0,K})$. So we see that it suffices to prove the following claim:

claim 1: Let E be an object in $F\text{-MIC}(\mathcal{X}_{0,K}, \overline{\mathcal{X}}'_{0,K})$. Then there exists an object \tilde{E} in $F\text{-MIC}(\mathcal{X}_{0,K}, \overline{\mathcal{X}}_{0,K})$ such that their restrictions to $F\text{-MIC}(\mathcal{X}_{0,K})$ are isomorphic.

For a subset I of $[1, s]$, let $\overline{\mathcal{X}}'_{0,I}$ be the open subscheme of \overline{X}_0 defined as $\overline{\mathcal{X}}'_{0,I} := \overline{X}_0 \setminus \{\prod_{j \in [1, s] \setminus I} t_j = 0\}$ and let $\overline{\mathcal{X}}'_{0,I}$ be the open formal subscheme of $\overline{\mathcal{X}}_0$ whose special fiber is equal to $\overline{\mathcal{X}}'_{0,I}$. Then

$$\begin{aligned} \mathfrak{U}_{0,I,\lambda} &:= \{x \in \overline{\mathcal{X}}'_{0,I,K} \mid \forall j \in I, |t_j(x)| \geq \lambda\} \\ &= \{x \in \overline{\mathcal{X}}_{0,K} \mid \forall j \in [1, s] \setminus I, |t_j(x)| = 1 \text{ and } \forall j \in I, |t_j(x)| \geq \lambda\} \\ &\quad (\lambda \in [0, 1) \cap \Gamma^*) \end{aligned}$$

gives a cofinal system of strict neighborhoods of $\mathcal{X}_{0,K}$ in $\overline{\mathcal{X}}'_{0,I,K}$. Let us put

$$A_{I,K} := \varinjlim_{\lambda \rightarrow 1} \Gamma(\mathfrak{U}_{0,I,\lambda}, \mathcal{O}_{\mathfrak{U}_{0,I,\lambda}}) = \Gamma(\overline{\mathcal{X}}'_{0,I,K}, j^\dagger \mathcal{O}_{\overline{\mathcal{X}}'_{0,I,K}})$$

(where $j^\dagger \mathcal{O}_{\overline{\mathcal{X}}'_{0,I,K}}$ denotes the sheaf of overconvergent sections for the formal smooth pair $(\mathcal{X}_0, \overline{\mathcal{X}}'_{0,I})$). Note that $A_{I,K}$ admits the canonical ring homomorphism $F^* : A_{I,K} \rightarrow A_{I,K}$ induced by $F : \overline{\mathcal{X}}_0 \rightarrow \overline{\mathcal{X}}_0$. We define the category $F\text{-MIC}(A_{I,K})$ as the category of pairs $((E, \nabla), \Psi)$, where (E, ∇) is a projective $A_{I,K}$ -module of finite rank endowed with an integrable connection $\nabla : E \rightarrow \bigoplus_{i=1}^d E dt_i$ and Ψ is an isomorphism $A \otimes_{F^*, A} (E, \nabla) \xrightarrow{=} (E, \nabla)$ (where $A \otimes_{F^*, A} -$ means the scalar extension by F^* as module endowed with integrable connection). Then we have the equivalence of categories

$$F\text{-MIC}(\mathcal{X}_{0,K}, \overline{\mathcal{X}}'_{0,I,K}) \xrightarrow{=} F\text{-MIC}(j^\dagger \mathcal{O}_{\overline{\mathcal{X}}'_{0,I,K}}) \xrightarrow{=} F\text{-MIC}(A_{I,K}).$$

Since we have $\overline{\mathcal{X}}'_{0,\emptyset} = \mathcal{X}_0$ and $\overline{\mathcal{X}}'_{0,[1,s]} = \overline{\mathcal{X}}_0$, we have

$$(3.11) \quad F\text{-MIC}(\mathcal{X}_{0,K}) \xrightarrow{=} F\text{-MIC}(A_{\emptyset,K}), \quad F\text{-MIC}(\mathcal{X}_{0,K}, \overline{\mathcal{X}}_{0,K}) \xrightarrow{=} F\text{-MIC}(A_{[1,s],K})$$

as particular cases. On the other hand, note that $\overline{\mathcal{X}}'_0 \setminus Z_0 = \bigcup_{i=1}^s \overline{\mathcal{X}}'_{0,\{i\}}$ and that $\overline{\mathcal{X}}'_{0,\{i\}} \cap \overline{\mathcal{X}}'_{0,\{i'\}} = \mathcal{X}_0$ for any $1 \leq i, i' \leq s, i \neq i'$. So we have the equivalences

$$(3.12) \quad \begin{aligned} & F\text{-MIC}(\mathcal{X}_{0,K}, \overline{\mathcal{X}}'_{0,K}) \\ & \xrightarrow{=} \text{fiber product of } F\text{-MIC}(\mathcal{X}_{0,K}, \overline{\mathcal{X}}'_{0,\{i\},K}) \rightarrow F\text{-MIC}(\mathcal{X}_{0,K}) \text{ (} i \in [1, s] \text{)} \\ & \xrightarrow{=} \text{fiber product of } F\text{-MIC}(A_{\{i\},K}) \rightarrow F\text{-MIC}(A_{\emptyset,K}) \text{ (} i \in [1, s] \text{)}. \end{aligned}$$

For $I' \subseteq I \subseteq [1, s]$, let us denote ‘the scalar extension functor’

$$F\text{-MIC}(A_{I,K}) \longrightarrow F\text{-MIC}(A_{I',K})$$

(which is induced by the canonical inclusion of the rings $A_{I,K} \hookrightarrow A_{I',K}$ by $A_{I',K} \otimes_{A_{I,K}} -$). Then the claim 1 is equivalent to the following claim:

claim 2: Let $((E_\emptyset, \nabla_\emptyset), \Psi_\emptyset)$ be an object in $F\text{-MIC}(A_{\emptyset,K})$ and for $1 \leq i \leq s$, let $((E_{\{i\}}, \nabla_{\{i\}}), \Psi_{\{i\}})$ be an object in $F\text{-MIC}(A_{\{i\},K})$ endowed with an isomorphism

$$f_{\{i\}} : A_{\emptyset,K} \otimes_{A_{\{i\},K}} ((E_{\{i\}}, \nabla_{\{i\}}), \Psi_{\{i\}}) \xrightarrow{=} ((E_\emptyset, \nabla_\emptyset), \Psi_\emptyset).$$

Then there exists an object $((E_{[1,s]}, \nabla_{[1,s]}), \Psi_{[1,s]})$ in $F\text{-MIC}(A_{[1,s],K})$ endowed with an isomorphism

$$f_{[1,s]} : A_{\emptyset,K} \otimes_{A_{[1,s],K}} ((E_{[1,s]}, \nabla_{[1,s]}), \Psi_{[1,s]}) \xrightarrow{=} ((E_\emptyset, \nabla_\emptyset), \Psi_\emptyset).$$

Before the proof of claim 2, we prove preliminary facts on commutative algebra.

claim 3: We have the following:

- (1) The homomorphism $F^* : A_{I,K} \longrightarrow A_{I,K}$ is flat for any $I \subseteq [1, s]$.
- (2) For $I, I' \subseteq [1, s]$, we have $A_{I,K} \cap A_{I',K} = A_{I \cup I',K}$. (Here the intersection is taken in $A_{\emptyset,K}$.)
- (3) Let $I \subseteq [1, s]$. Then, For any projective $A_{\emptyset,K}$ -module of finite rank E_\emptyset , there exists a projective $A_{I,K}$ -module of finite rank E_I endowed with an isomorphism $f_I : A_{\emptyset,K} \otimes_{A_{I,K}} E_I \xrightarrow{=} E_\emptyset$. If there are two such data $(E_I, f_I), (E'_I, f'_I)$, there exists an isomorphism $g : E_I \xrightarrow{=} E'_I$ such that the composite

$$E_\emptyset \xrightarrow{f_I^{-1}} A_{\emptyset,K} \otimes_{A_{I,K}} E_I \xrightarrow{\text{id} \otimes g} A_{\emptyset,K} \otimes_{A_{I,K}} E'_I \xrightarrow{f'_I} E_\emptyset$$

is the identity map.

We prove the claim 3. For $n := (n_1, \dots, n_d) \in \mathbb{Z}^d$ and $I \subseteq [1, d]$, we put $n_I := \sum_{i \in I} n_i$. Then, by definition, the ring $A_{I,K}$ has the following concrete description:

$$A_{I,K} := \left\{ \sum_{n \in \mathbb{Z}^s \times \mathbb{N}^{d-s}} a_n t^n \mid a_n \in K, \exists \lambda \in [0, 1), \forall J \subseteq I, |a_n| \lambda^{n_J} \rightarrow 0 (|n| \rightarrow \infty) \right\}.$$

(See [11, 3.1.7] for example.) Using this description, it is easy to see that $A_{I,K}$ is freely generated by $\{t^n\}_{n=(n_1, \dots, n_d), 0 \leq n_i \leq q-1}$ when it is regarded as an $A_{I,K}$ -module via F^* . So we have (1). (2) follows from the above description and the inequality

$$|a_n| \lambda^{n_J/2} \leq \max\{|a_n| \lambda^{n_{J \cap I}}, |a_n| \lambda^{n_{J \setminus I}}\}$$

for any $J \subseteq I \cup I'$. To prove (3), we put $R_I := O_K\{t_i (1 \leq i \leq d), t_j^{-1} (j \in [1, s] \setminus I)\}$ and let A_I be the weak completion of $R_I[t_j^{-1} (j \in I)]$ over (R_I, pR_I) in the sense of Monsky-Washnitzer [15]. (A_I is a w.c.f.g. algebra over (R_I, pR_I) by definition.) Then, using [15, 2.2, 2.3], we see easily the equality $A_{I,K} = \mathbb{Q} \otimes_{\mathbb{Z}} A_I$ by direct calculation. Also, by [5, Théorème 3], $(\text{Spec } A_I, \text{Spec } A_I/pA_I)$ is a Henselian couple in the sense of [7, 18.5.5]. Then we have the assertion (3) by [4, Corollaire 1 du Théorème 3] and [4, Lemme in p.573] (see also the proof of [6, Proposition 3]).

Now we prove the claim 2, using claim 3. Note that it suffices to prove the following claim $(*)_j$ by induction on j ($1 \leq j \leq s$):

$(*)_j$: With the assumption of the claim 2, there exists an object $((E_{[1,j]}, \nabla_{[1,j]}), \Psi_{[1,j]})$ in $F\text{-MIC}(A_{[1,j],K})$ endowed with an isomorphism

$$f_{[1,j]} : A_{\emptyset,K} \otimes_{A_{[1,j],K}} ((E_{[1,j]}, \nabla_{[1,j]}), \Psi_{[1,j]}) \xrightarrow{=} ((E_{\emptyset}, \nabla_{\emptyset}), \Psi_{\emptyset}).$$

$(*)_1$ is true by assumption. Now we assume $(*)_{j-1}$ and prove $(*)_j$. (The argument in the following is inspired by that in [6, I].) By (3) of claim 3, there exists a projective $A_{[1,j],K}$ -module $E_{[1,j]}$ of finite rank endowed with an isomorphism $f_{[1,j]} : A_{\emptyset,K} \otimes_{A_{[1,j],K}} E_{[1,j]} \xrightarrow{=} E_{\emptyset}$. Then, since both $A_{[1,j-1],K} \otimes_{A_{[1,j],K}} E_{[1,j]}$ and $E_{[1,j-1]}$ are isomorphic to E after we apply $A_{\emptyset,K} \otimes_{A_{[1,j-1],K}} -$, they are canonically isomorphic. By the same reason, $A_{\{j\},K} \otimes_{A_{[1,j],K}} E_{[1,j]}$ and $E_{\{j\}}$ are canonically isomorphic. Since $E_{[1,j]}$ is projective, we can regard $E_{[1,j]}$ as a submodule of $E_{[1,j-1]} \cap E_{\{j\}}$ (the intersection is taken in E_{\emptyset}). Moreover, by claim 3 (2) and the projectivity of $E_{[1,j]}$, we see that $E_{[1,j]}$ is equal to $E_{[1,j-1]} \cap E_{\{j\}}$. Then, using claim 3 (1), we see that $A_{[1,j],K} \otimes_{F^*, A_{[1,j],K}} E_{[1,j]}$ is equal to $(A_{[1,j-1],K} \otimes_{F^*, A_{[1,j-1],K}} E_{[1,j-1]}) \cap (A_{\{j\},K} \otimes_{F^*, A_{\{j\},K}} E_{\{j\}})$. Then we can define $\nabla_{[1,j]} : E_{[1,j]} \rightarrow \bigoplus_{i=1}^d E_{[1,j]} dt_i$ and $\Psi_{[1,j]} : A_{[1,j],K} \otimes_{F^*, A_{[1,j],K}} (E_{[1,j]}, \nabla_{[1,j]}) \rightarrow (E_{[1,j]}, \nabla_{[1,j]})$ simply by

$$\nabla_{[1,j]} := \nabla_{[1,j-1]}|_{E_{[1,j]}} = \nabla_{\{j\}}|_{E_{[1,j]}},$$

$$\Psi_{[1,j]} := \Psi_{[1,j-1]}|_{A_{[1,j],K} \otimes_{F^*, A_{[1,j],K}} E_{[1,j]}} = \Psi_{\{j\}}|_{A_{[1,j],K} \otimes_{F^*, A_{[1,j],K}} E_{[1,j]}}.$$

Then it is easy to see that $((E_{[1,j]}, \nabla_{[1,j]}), \Psi_{[1,j]})$ satisfies the required condition in $(*)_j$. So we have proved the claim 2 (hence the claim 1) and therefore the proof of the theorem is now finished. \square

4 An application

In this section, assume that k is perfect. Let X be a connected smooth scheme over k and let $\pi_1(X)$ be the fundamental group of X (we omit to write the base point). Denote by $\text{Rep}_{K^\sigma}(\pi_1(X))$ the category of finite-dimensional continuous representations of $\pi_1(X)$ over K^σ , where K^σ denotes the fixed field of K by σ . In [3], Crew proved that there exists the canonical equivalence

$$(4.1) \quad G : \text{Rep}_{K^\sigma}(\pi_1(X)) \xrightarrow{\cong} F\text{-Isoc}(X)^\circ,$$

where $F\text{-Isoc}(X)^\circ$ denotes the category of unit-root convergent F -isocrystals on X , that is, the category of convergent F -isocrystals on X satisfying certain condition called ‘unit-root condition’. (For the definition, see [3].)

Assume now that the above X is enclosed to a pair (X, \overline{X}) and let S be the set of all the discrete valuations of $k(X)$ centered on $\overline{X} \setminus X$. (As for the notion of the center of valuation, see [12] or [20].) For $v \in S$, let us denote the inertia subgroup of $k(X)_v$ ($:=$ the completion of $k(X)$ with respect to v) by I_v . Then we have the natural homomorphism $I_v \longrightarrow \pi_1(X)$ which is well-defined up to conjugate. Then we define the subcategory of $\text{Rep}_{K^\sigma}(\pi_1(X))$ with finite local monodromy along S by

$$\text{Rep}_{K^\sigma}^S(\pi_1(X)) := \{\rho \in \text{Rep}_{K^\sigma}(\pi_1(X)) \mid \forall v \in S, \rho|_{I_v} \text{ has finite image}\}.$$

Tsuzuki has shown in [18] that, in the case where X is a curve, the functor G of Crew restricts to the equivalence of categories

$$\text{Rep}_{K^\sigma}^S(\pi_1(X)) \xrightarrow{\cong} F\text{-Isoc}^\dagger(X, \overline{X})^\circ,$$

where $F\text{-Isoc}^\dagger(X, \overline{X})^\circ$ denotes the category of unit-root overconvergent F -isocrystals on (X, \overline{X}) , which is the category $F\text{-Isoc}^\dagger(X, \overline{X}) \cap F\text{-Isoc}(X)^\circ$. In higher dimensional case, Kedlaya [13, 2.3.7, 2.3.9] proved the following result, based on a result of Tsuzuki in [19]: Let $\text{Rep}_{K^\sigma}(\pi_1(X))'$ be the subcategory of $\text{Rep}_{K^\sigma}(\pi_1(X))$ consisting of the representations ρ such that, for some finite morphism $\varphi : \overline{Y} \longrightarrow \overline{X}$ with $Y := \varphi^{-1}(X) \longrightarrow X$ finite etale, the restriction of ρ to I_v is trivial for any discrete valuation v of $k(Y)$ centered on $\overline{Y} \setminus Y$. (He calls such a representation ‘potentially unramified’.) Then he has shown that the functor G of Crew restricts to the equivalence of categories

$$\text{Rep}_{K^\sigma}(\pi_1(X))' \xrightarrow{\cong} F\text{-Isoc}^\dagger(X, \overline{X})^\circ.$$

Note that we have the following proposition, which implies that Kedlaya’s result is actually a generalization of Tsuzuki’s one:

Proposition 4.1. *With the above notation (with X of arbitrary dimension), we have $\text{Rep}_{K^\sigma}^S(\pi_1(X)) = \text{Rep}_{K^\sigma}(\pi_1(X))'$.*

Proof. First let ρ be an object in $\text{Rep}_{K^\sigma}(\pi_1(X))'$. Let us take $\varphi : \overline{Y} \rightarrow \overline{X}$, Y as above and let $v \in S$. Let v' be a valuation which extends v (hence centered on $\overline{Y} \setminus Y$). Then $\rho|_{I_{v'}}$ is trivial. Since $I_v/I_{v'}$ is finite, we see that $\rho|_{I_v}$ has finite image and so $\rho \in \text{Rep}_{K^\sigma}^S(\pi_1(X))$.

On the other hand, let ρ be an object in $\text{Rep}_{K^\sigma}^S(\pi_1(X))$. We may assume that ρ has the form $\pi_1(X) \rightarrow GL_n(O_{K^\sigma})$ (where O_{K^σ} is the ring of integers of K^σ). Let $\overline{\rho}$ be the composite $\pi_1(X) \rightarrow GL_n(O_{K^\sigma}) \twoheadrightarrow GL_n(O_{K^\sigma}/2pO_{K^\sigma})$ and let $\varphi : Y \rightarrow X$ be the finite etale morphism corresponding to the subgroup $\text{Ker}(\overline{\rho}) \subseteq \pi_1(X)$. Let $\overline{Y} \rightarrow \overline{X}$ be the normalization of \overline{X} in $k(Y)$, which will be also denoted by φ . Let v be a discrete valuation of $k(Y)$ centered on $\overline{Y} \setminus Y$. Then, by definition of ρ , $\rho|_{I_{v|_{k(X)}}$ has finite image. Hence so is $\rho|_{I_v}$. On the other hand, the image of $\rho|_{I_v}$ is contained in $\text{Ker}(GL_n(O_{K^\sigma}) \rightarrow GL_n(O_{K^\sigma}/2pO_{K^\sigma}))$ and $\text{Ker}(GL_n(O_{K^\sigma}) \rightarrow GL_n(O_{K^\sigma}/2pO_{K^\sigma}))$ contains no nontrivial finite subgroup. So we can conclude that $\rho|_{I_v}$ is trivial. Hence we have $\rho \in \text{Rep}_{K^\sigma}(\pi_1(X))'$. \square

One drawback in the above-mentioned result of Kedlaya in higher-dimensional case is that the number of valuations which we should look at is infinite. In this section, we give an alternative formulation of ‘the subcategory of $\text{Rep}_{K^\sigma}(\pi_1(X))$ with finite local monodromy’ which looks at only finitely many discrete valuations of $k(X)$ and which is still equivalent (via G) to $F\text{-Isoc}(X, \overline{X})^\circ$. To do so, let us take a separable alteration $f : \overline{X}' \rightarrow \overline{X}$ such that $(X' := f^{-1}(X), \overline{X}')$ is a smooth pair. (Such f exists by de Jong’s theorem [8, 4.1]). Let $\overline{X}' \setminus X' = \bigcup_{i=1}^r Y_i$ be the decomposition of $\overline{X}' \setminus X'$ (with reduced closed subscheme structure) into irreducible components, let v_i ($1 \leq i \leq r$) be the discrete valuation on $k(X')$ corresponding to Y_i and let us put $S' := \{v_i|_{k(X)} \mid 1 \leq i \leq r\}$. (Then S' is a finite subset of S .) We define the subcategory of $\text{Rep}_{K^\sigma}(\pi_1(X))$ with finite local monodromy along S' by

$$\text{Rep}_{K^\sigma}^{S'}(\pi_1(X)) := \{\rho \in \text{Rep}_{K^\sigma}(\pi_1(X)) \mid \forall v \in S', \rho|_{I_v} \text{ has finite image}\}.$$

Then the main theorem in this section is as follows:

Theorem 4.2. *Let the notations be as above. Then the equivalence G of Crew (4.1) restricts to the equivalence*

$$\text{Rep}_{K^\sigma}^{S'}(\pi_1(X)) \xrightarrow{\sim} F\text{-Isoc}^\dagger(X, \overline{X})^\circ.$$

In particular, we have the equality $\text{Rep}_{K^\sigma}^S(\pi_1(X)) = \text{Rep}_{K^\sigma}^{S'}(\pi_1(X))$.

Proof. We prove the following two claims:

- (1) $\text{Rep}_{K^\sigma}^S(\pi_1(X)) \supseteq G^{-1}(F\text{-Isoc}^\dagger(X, \overline{X})^\circ)$.
- (2) $\text{Rep}_{K^\sigma}^{S'}(\pi_1(X)) \subseteq G^{-1}(F\text{-Isoc}^\dagger(X, \overline{X})^\circ)$.

Since we have $\text{Rep}_{K^\sigma}^S(\pi(X)) \subseteq \text{Rep}_{K^\sigma}^{S'}(\pi(X))$ by definition, the above two claims imply the theorem.

Let us prove (1). (Here we give a proof which is different from that in [13, 2.3.7, 2.3.9].) Let \mathcal{E} be an object in $F\text{-Isoc}^\dagger(X, \overline{X})^\circ$ and put $\rho := G^{-1}(\mathcal{E})$. By [19, 1.3.1], there exists a separable alteration $\varphi : \overline{X}'' \rightarrow \overline{X}$ such that, if we put $X'' := \varphi^{-1}(X)$, the restriction of \mathcal{E} to $F\text{-Isoc}^\dagger(X'', \overline{X}'')^\circ$ extends to an object in $F\text{-Isoc}(\overline{X}'')^\circ$. This implies (via Crew's equivalence for \overline{X}'') that the restriction of ρ to $\pi_1(X'')$ factors through $\pi_1(\overline{X}'')$. Now let us take a discrete valuation v of $k(X)$ centered on $\overline{X} \setminus X$ and let v' an extension of v to $k(X'')$ (hence centered on $\overline{X}'' \setminus X''$). Let z be the center of v' and let $O_{v'}$ be the valuation ring of $k(X')_{v'}$. Then the homomorphism

$$(4.2) \quad \pi_1(\text{Spec } k(X'')_{v'}) \rightarrow \pi_1(X'') \rightarrow \pi_1(\overline{X}'')$$

factors as

$$(4.3) \quad \pi_1(\text{Spec } k(X'')_{v'}) \rightarrow \pi_1(\text{Spec } O_{v'}) \rightarrow \pi_1(\text{Spec } \mathcal{O}_{\overline{X}'', z}) \rightarrow \pi_1(\overline{X}'').$$

Since the restriction ρ to $\pi_1(\text{Spec } k(X'')_{v'})$ factors through (4.2)=(4.3), we see that $\rho|_{I_{v'}}$ is trivial. Hence $\rho|_{I_v}$ is finite and so we have $\rho \in \text{Rep}_{K^\sigma}^S(\pi(X))$. So the proof of (1) is finished.

Let us prove (2). Let us take $\rho \in \text{Rep}_{K^\sigma}^{S'}(\pi_1(X))$. We may assume that ρ has the form $\pi_1(X) \rightarrow GL_n(O_{K^\sigma})$. Let $\overline{\rho}$ be the composite $\pi_1(X) \rightarrow GL_n(O_{K^\sigma}) \rightarrow GL_n(O_{K^\sigma}/2pO_{K^\sigma})$ and let $X'' \rightarrow X'$ be the finite etale morphism corresponding to the subgroup $\text{Ker}(\overline{\rho}|_{\pi_1(X')}) \subseteq \pi_1(X')$. Let $g : \overline{X}'' \rightarrow \overline{X}'$ be the normalization of \overline{X}' in $k(X'')$ and let us take a separable alteration $h : \overline{X}''' \rightarrow \overline{X}''$ such that the pair $(X''' := h^{-1}(X''), \overline{X}''')$ is a smooth pair. Then ρ induces the morphism

$$(4.4) \quad \rho|_{\pi_1(X''')} : \pi_1(X''') \rightarrow \text{Ker}(GL_n(O_{K^\sigma}) \rightarrow GL_n(O_{K^\sigma}/2pO_{K^\sigma})).$$

Let us put $\overline{X}''' \setminus X''' = \bigcup_{i=1}^a Y'_i \cup \bigcup_{i=a+1}^b Y'_i$, where Y'_i 's ($1 \leq i \leq a$) are the irreducible components of $\overline{X}''' \setminus X'''$ with $\text{codim}(g \circ h(Y'_i), \overline{X}') = 1$ and Y'_i 's ($a+1 \leq i \leq b$) are the irreducible components of $\overline{X}''' \setminus X'''$ with $\text{codim}(g \circ h(Y'_i), \overline{X}') \geq 2$. Let us put $Z := g \circ h(\bigcup_{i=a+1}^b Y'_i)$. Then $g \circ h$ induces the morphism of smooth pairs $g \circ h : (X''', \overline{X}''' \setminus (g \circ h)^{-1}(Z)) \rightarrow (X', \overline{X}' \setminus Z)$. For $1 \leq i \leq a$, let v'_i be the discrete valuation of $k(X''')$ corresponding to Y'_i . Then, by definition, $v'_i|_{k(X)} \in S'$. Hence $\rho|_{I_{v'_i|_{k(X)}}}$ has finite image and so is $\rho|_{I_{v'_i}}$. On the other hand, by (4.4), the image of $\rho|_{I_{v'_i}}$ is contained in $\text{Ker}(GL_n(O_{K^\sigma}) \rightarrow GL_n(O_{K^\sigma}/2pO_{K^\sigma}))$. Since $\text{Ker}(GL_n(O_{K^\sigma}) \rightarrow GL_n(O_{K^\sigma}/2pO_{K^\sigma}))$ contains no nontrivial finite subgroup, $\rho|_{I_{v'_i}}$ is trivial for any $1 \leq i \leq a$. So we see that $\rho|_{\pi_1(X''')}$ factors through $\pi_1(\overline{X}''' \setminus (g \circ h)^{-1}(Z))$. This implies (via Crew's equivalence) that the restriction of $G(\rho) \in F\text{-Isoc}(X)^\circ$ to $F\text{-Isoc}(X''')^\circ$ extends to $F\text{-Isoc}(\overline{X}''' \setminus (g \circ h)^{-1}(Z))^\circ$ (hence to $F\text{-Isoc}^\dagger(X''', \overline{X}''' \setminus (g \circ h)^{-1}(Z))^\circ$). Then, by [2, 2.1.2] (applied to $g \circ h$), we see that the restriction of $G(\rho)$ to $F\text{-Isoc}(X')^\circ$

extends to $F\text{-Isoc}^\dagger(X', \overline{X}' \setminus Z)^\circ$. By Theorem 3.1, it extends to $F\text{-Isoc}^\dagger(X', \overline{X}')^\circ$, since $\text{codim}(Z, \overline{X}') \geq 2$. Then, again by [2, 2.1.2] (applied to f), we see that $G(\rho)$ extends to $F\text{-Isoc}^\dagger(X, \overline{X})^\circ$. So we have proved (2). \square

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